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A RELATION BETWEEN THE CIRCULAR AND THE PROJECTIVE TRANSFORMATIONS OF THE PLANE*

By Edward Kasner

1. Introduction. Among the transformations of the plane, those which play the most important rôle in modern geometry are the two mentioned in the title.† The projective transformations (otherwise homographies or collineations) are characterized among point transformations by the property that straight lines are converted into straight lines; and are represented, in Cartesian coordinates, as follows:

$$(H) \qquad x' = \frac{a_1 x + b_1 y + c_1}{a_3 x + b_3 y + c_3}, \quad y' = \frac{a_2 x + b_2 y + c_2}{a_3 x + b_3 y + c_3};$$

thus constituting a continuous group of eight parameters. The circular transformations, on the other hand, convert circles into circles, and are represented most conveniently in terms of the complex variable. They divide into two species, those of the first taking the form

$$(\Gamma_1) \qquad z' = \frac{\alpha z + \beta}{\gamma z + \delta} \qquad (z = x + iy),$$

while those of the second (distinguished by the reversal of angles) take the form

$$(\Gamma_2) \qquad z' = \frac{\alpha z + \beta}{\gamma \bar{z} + \delta} \qquad (z = x - iy).$$

The totality of circular transformations constitutes then a mixed group of six real parameters, composed of two continuous systems.

The simple relation established in the present paper shows how either type of transformation may be obtained from the other type. It is of course useless to seek for a (continuous) one to one correspondence between the two classes, since the number of parameters involved is not the same. But the effect of the relation in question may be roughly described as the establishment of a correspondence between the homographies (H) on the one hand, and, on the other, the combinations (Γ, O), each composed of a circular transformation and a point.

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† See, for example, Klein's *Erlanger Programm* and *Höhere Geometrie.*

The correspondence, however, breaks down for certain special transformations — a circumstance which is to be expected in view of the fact that in projective geometry the infinite region of the plane is regarded as a straight line, while in inversion geometry it is equivalent to a point. The exceptional transformations are indeed those which convert the infinite region into itself. We shall assume then in the following, unless stated otherwise, that H is not an affinity, and Γ is not a similarity. Consequently H has in each plane two finite points, the first and the second focus, the centres respectively of the corresponding congruent and symmetric pencils of rays;[*] and Γ has, in each plane, a unique finite point, the pole, which corresponds to the infinite point in the other plane.[†]

2. From Circular to Projective Transformations. We proceed now to the relation in question. Consider a circular transformation Γ and a finite point O which does not coincide with the pole of Γ. Denote the corresponding point, necessarily finite, in the second plane, by O'. Any circle through O is converted by Γ into a circle through O'. But to each of these circles there corresponds a unique point, its center. In this way, we obtain a certain transformation, H, not coinciding with Γ, whose effect upon any point P is described as follows: Construct the circle γ passing through O and having P for center; this is converted by Γ into a circle γ' passing through O'; the center of γ' is the point P' required.

We now prove that the transformation H, so obtained, is a homography, by showing that any straight line is converted into a straight line. Consider any line l. The circles passing through O and having their centers on l, form a pencil. These circles are converted by Γ into circles through O' which also form a pencil. Therefore the centers lie on a line l'. This proves the result stated.

The construction for l' may be described more simply as follows: Construct the image of O with respect to l, say P; next the point P' into which P

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[*] H. F. S. Smith, The focal properties of homographic figures, § 5, *Proc. Lond. Math. Soc.* (1869) or *Collected Mathematical Papers*, vol. 1, p. 549. According to the historical note at the end of this memoir, Magnus and Chasles in their investigations considered only the first focus.

[†] Möbius, Theorie der Kreisverwandtschaften in rein geometrischer Darstellung, §6, *Gesammelte Werke*, vol. 2.

is converted by Γ; finally the mid-line of the points O' and P'; this is the required line l'.

This construction fails, however, if the line l passes through O; but in this case the line l' passes through O', and is easily obtained by taking any circle γ tangent to l at O; the line l' is then tangent to the circle γ' into which Γ converts γ. It follows that the angle of any two lines passing through O, is equal to the angle of the corresponding lines through O', being reversed in direction however if Γ is of the second species. Hence the point O is a focus of H. The result obtained may be stated:

THEOREM I. *To a circular transformation* Γ, *and a point* O *which is not its pole, corresponds a unique collineation* H. *The point* O *is the first or second focus of* H *according as* Γ *is of the first or second species.**

3. From Projective to Circular Transformations. The H obtained of course depends, in general, upon both Γ and O; so that a single circular transformation gives rise to ∞^2 collineations,† corresponding to the ∞^2 positions of O. It seems plausible that we obtain, by the above construction, the totality of collineations, or at least all those possessing foci.‡ The exact theorem follows:

THEOREM II. *To any homography* H *which is not an affinity, there correspond two combinations* (Γ, O), *composed of a circular transformation and a point, each of which gives rise to* H. *In one of these,* (Γ_1, F_1), *the transformation is of the first species, and the point is the first focus of* H; *while in the other,* (Γ_2, F_2), *the transformation is of the second species and the point is the second focus of* H.

For the proof, we note first that the point O must be a focus of H. Consider then either focus F, and the corresponding point F' in the other plane. Take any three points A, B, C, such that no three of the four points F, A, B, C are collinear; and let the corresponding points be A', B', C'. Consider the circles α, β, γ passing through F and having A, B, C for centers; and similarly the circles α', β', γ' passing through F' and with centers at

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* For the exceptional case noted in the introduction, when Γ is a similarity S, the first part of the result still holds. The homography obtained is then simply S. Even the second part may be said to hold, since in the case of a similarity any point has the property of a focus.

† It may be remarked that these do *not* form a group.

‡ In the case of any affinity no focus exists, except when it is merely a similarity S; in the latter case, any point may be considered as a focus, so that the corresponding combinations are (S, O), where O is any point.

A', B', C'. We have thus in each plane a set of three circles of a bundle. Since the angle of any two circles of either bundle is equal to the angle between the lines drawn from the focus to the centers, it follows that corresponding angles in the two bundles are equal. Hence there exists a unique circular transformation Γ which converts α, β, γ into α', β', γ' respectively. This transformation Γ together with the point F' gives rise, by Theorem I, to a unique homography. It is easily seen that the homography so obtained converts F, A, B, C into F', A', B', C', and therefore coincides with the given collineation H. This completes the proof of (II).

The effect of Γ may be described as follows: Consider any point P; construct the mid-line l of F and P; next the line l' into which l is converted by H; finally the image of O' with respect to l'; this is the point P' which corresponds, by Γ, to the original point P.

4. Analytical Representation of the Relation.

By a proper choice of the system of axes in each plane, it is possible to reduce the general collineation H to the following metrically normal form:*

$$x' = \frac{cc'}{x}, \quad y' = \frac{c'y}{x}. \tag{1}$$

The origin in each plane is the point midway between the two foci, and the axis of abscissas is the line joining the foci; the axis of ordinates is then the vanishing line. In the first plane the foci are

$$F_1: \quad (-c, 0); \qquad F_2: \quad (c, 0). \tag{2}$$

The corresponding circular transformations are obtained by following out, step by step, the construction given at the end of the preceding section; the results are:

$$\Gamma_1: \quad z' = c' \frac{z + 3c}{z - c}, \tag{3}$$

$$\Gamma_2: \quad z' = -c' \frac{\bar{z} - 3c}{\bar{z} + c}. \tag{4}$$

We note from these normal forms, that *the pole of* Γ_1 *is* F_2 *the second focus of* H, *and the pole of* Γ_2 *is* F_1 *the first focus of* H. The coordinate system may therefore be defined anew, with reference to the combination (Γ, F), as follows: the origin is the point midway between F and the pole of Γ, and the axis of abscissas the line joining the same points.

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* *Cf.* §15 of Smith's memoir, cited above.

We now inquire concerning those points upon which H and Γ have the same effect. Considering in the first place Γ_1, we find from (1) and (3) that the points in question satisfy the condition

$$\frac{c'(c+iy)}{x} = \frac{c'(x+iy+3c)}{x+iy-c},$$

which reduces to

$$(x+c)^2 + y^2 = 0\cdot$$

This represents the null circle whose center is F. Similarly, comparing (4) and (1), we obtain

$$\frac{c'(c+iy)}{x} = \frac{-c'(x-iy-3c)}{x-iy+c},$$

which reduces to

$$(x-c)^2 + y^2 = 0,$$

and represents the null circle with center at F_2.

If the homography H corresponds to the combination (Γ, O), then the only real point upon which H and Γ have the same effect is the point O. The imaginary points having the property in question lie on the minimal lines through O.

It is easy to determine the direct relation of the combinations $(\Gamma_1\ F_1)$ and (Γ_2, F_2), that is, the relation in a form independent of the homography H which both define. For this purpose, we observe that (4) may be decomposed into the following three transformations:

$$w = -z, \quad w' = c'\frac{w+3c}{w-c}, \quad z' = -\bar{w}',$$

of which the first represents central symmetry with respect to the origin in the first plane, the second is simply Γ_1, and the third represents symmetry with respect to the axis of ordinates in the second plane. Hence the result:

To any combination (Γ_1, F_1), consisting of a circular transformation and a point, corresponds a second combination (Γ_2, F_2) as follows: the point F_2 is the pole of Γ_1; and Γ_2 is generated by central symmetry with respect to the midpoint of F_1, F_2, followed by Γ_1, followed by symmetry with respect to the midline of the points F_1', F_2'; where F_1' is the point into which Γ_1 transforms F_1, *and* F_2' *is the center of the circle into which* Γ_1 *transforms the circle passing through* F_1 *with center* F_2. *The transformations* Γ_1, Γ_2 *are opposite in species, and the correspondence considered is mutual.*

5. Applications to Invariant Relations. Consider any system of circles passing through a common point O and having their centers on an algebraic curve c_n of the nth order. *Any circular transformation converts such a system into a system of the same kind;* for the centers undergo a projective transformation and hence still lie on a curve c_n' of nth order. In the case $n = 1$, the system is merely a pencil of circles, and the theorem gives a well-known result. For $n = 2$, the theorem announces that a system of circles passing through a common point and with centers lying on a circle, or more generally any conic, is converted into a system with centers lying on a conic.

Any projective relation among the centers of a set of circles of a bundle, corresponds thus to a relation among the circles which is invariant under the group of circular transformations. Consider any number of such circles $\gamma_1, \gamma_2, \cdots$ passing through the point O; let the centers be P_1, P_2, $\cdots$ *Then any projective absolute invariant of the points* P_1, P_2, $\cdots$, *or more generally of these points together with* O, *gives an absolute invariant* (with respect to circular transformations) *of the circles* $\gamma_1, \gamma_2, \cdots$.

For example, consider three circles touching each other at a point. The three centers and the point of contact are collinear. The anharmonic ratio of these four points is then an absolute invariant of the three circles. This is perhaps the simplest example of a circular absolute invariant which is not expressible in terms of the various angles of intersection. If we take the more general case of four circles through a point, then the anharmonic ratio of the lines joining say P_1 to P_2, P_3, P_4, O is an absolute invariant of the circles, which can in general be expressed in terms of angles.

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